A NOTE ON A GENERALIZATION OF THE GAUSS-LUCAS THEOREM

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Abstract. Given a set of points $z_1, z_2, \ldots, z_n$, in the complex plane, we define incomplete polynomial as the polynomial which have zeros at these points except one of them. The classical result Gauss - Lucas theorem [5] on the location of zeros of a polynomial and it's derivative was extended to the linear combination of incomplete polynomials by J.L.Diaz - Barrero [3]. In this paper, we extend Specht theorem [5] and results proved

by A. Aziz [1] to the class of linear combination of incomplete polynomials. Moreover our results refines the results proved in [3].

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1. Introduction

Mathematical models are created by translating various scientific findings and investigations into mathematical terms. Solving these models may result in some degree of difficulty in solving algebraic polynomial equations. There is no such method available for completing the same work for polynomials of higher degree. The exact computation of zeros of polynomials of degree at most four is made possible by algorithms having been designed for such polynomials. This accomplishment is impossible, or to put it another way, polynomial equations of degree five or higher cannot be solved by radicals.

This crucial mathematical milestone was caused by the revolutionary algebraic discoveries of N. H. Abel and E. Galois in the first quarter of the nineteenth century. This and substantial uses of zero bounds in fields of science including stability theory, mathematical biology, communication theory, and computer engineering made it intriguing to pinpoint the appropriate areas in the complex plane that a given polynomial's zeros fall within. Let f denote the linear space of all polynomials of degree n over the field C of complex numbers, f_n denotes the collection of all $_{+}$ monic polynomials of degree n in f , where n is positive integer, $K(f\,)$ denotes the convex hull of zeros of $f\,(z)$ and R^{n} be set of all $n-$tuples $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ of positive real numbers with $\gamma_1 + \gamma_2 + ... + \gamma_n = 1$. The classical Gauss - Lucas theorem which gives a relationship between the zeros and critical points of a polynomial, states the following.

Theorem 1.1. *The convex hull $K(f\,)$ of the zeros of polynomial $f\,(z)$ contains all its critical points.*

Furthermore, if none of the zeros of $f\,(z)$ is simple and are not collinear then no critical point of $f\,(z)$ lie on the boundary of $K(f\,)$. Also if z^* is a simple zero of $f\,(z)$ and

$f\,'(z^*)$ = 0, thus, by continuity their exists a neighborhood of z^* which does not contain a critical point of $f\,(z)$. A method of constructing such a neighborhood is given by Krawtchouk, which after words used to deduce refinement of Gauss-Lucas theorem. In fact he proved the following result.

Theorem 1.2. *Let $z_1, z_2, . . . , z_n$, be n, not necessarily distinct, be the zeros of $f\,(z)$ with a simple zero at z_1. Define*

$$\omega_v = \frac{z_v + (n-1)z_1}{n}, \quad v = 2, 3, ..., n.$$

If M is an open disc or open half plane that contains z_1 but none of the points $\omega_2, \omega_3, . . . , \omega_n$. Then M is devoid of critical points of the polynomial $f\,(z)$.

Moreover, Specht [6] using Theorem 1.2 and proved a result on the relationship between zeros and critical points which is also a refinement of Gauss-Lucas theorem. In fact he proved the following interesting result.

Theorem 1.3. *Let $f\,(z)$ be a polynomial of degree n with zeros $z_1, z_2, . . . , z_n$, then the convex hull $K(f\,(z))$ of $n^2 - n$ points*

$$\omega_{v\mu} = \frac{z_v + (n-1)z_\mu}{n}, \quad v, \mu \in \{1, 2, ..., n\}, v \neq \mu$$

contains all critical points of the polynomial $f\,(z)$.

For the more research and development in this direction, one can see some papers

([7], [4] , [8]) and can consult a book of Rahman and Schmeisser [6].

Let $f(z)$ be a polynomial of degree n having all its zeros in the disk $|z-z^*|\le 1$ and $f(z^*) = 0$, then the famous Sendov's conjecture [3, p.224], says that the closed disk $|z-z^*|\le 1$ contains a critical point of $f(z)$. The conjecture has been proved for the polynomials of degree at most eight[2]. Also, the conjecture is true for some special class of polynomials such as the polynomials having a zero at the origin and the polynomials having all their zeros on $|z|=1$, as shown in [2]. However, the general version is still unproved. A Aziz [1] proved the following results regarding the relationship between the zeros and critical points of a polynomial.

Theorem 1.4. *If $f(z)$ is a polynomial of degree n and ω is a zero of $f'(z)$, then for every given real or complex number α, $f(z)$ has at least one zero in the region*

$$|w-\frac{\alpha+z}{2}|\le|\frac{\alpha-z}{2}|.$$

Taking $\alpha=0$ in Theorem 1.4 and noting that $|w-\frac{z}{2}|\le|\frac{z}{2}| \Rightarrow |w-z|\le|z|$, A Aziz [1] deduced the following result.

Theorem 1.5. *If all the zeros of a polynomial $f(z)$ of degree n lie in $D = \{z \in C : |z| \le 1\}$, and ω is zero of $f'(z)$, then $f(z)$ has at least one zero in both the circles*

$$|w-\frac{z}{2}|\le|\frac{1}{2}| \quad and \quad |w-z|\le 1.$$

Definition 1.6. (Incomplete polynomials): Let $z_1, z_2, \ldots, z_n$ be n, not necessary distinct, complex numbers. The incomplete polynomial of degree $n-1$ associated with $z_1, z_2, \ldots, z_n$, are the polynomials $g_k(z)$, $1\le k\le n$, defined by

$$g_k(z)=\prod_{j=1, j\ne k}^{n}(z-z_j) \qquad (1.1)$$

Let $f_n(z)$ be a monic polynomial with zeros at $z_1, z_2, \ldots, z_n$, then,

$$f_n(z)=\prod_{k=1}^{n}(z-z_k) \qquad (1.2)$$

then the derivative of $f_n(z)$ is given by

$$\frac{1}{n}f_n'(z)=\frac{1}{n}\sum_{k=1}^{n}g_k(z) \qquad (1.3)$$

Thus, the derivative $\frac{1}{n}f_n'(z)$ is a linear convex combination of the incomplete polynomials $g_k(z)$, $k = 1, 2, \ldots, n$.

Now, let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be non - negative real numbers such that

$$\sum_{k=1}^{n}\gamma_k=1 \qquad (1.4)$$

The convex combination of $g_k(z)$, $k = 1, 2, \ldots, n$ with respect to $\gamma_1, \gamma_2, \ldots, \gamma_n$ is given by

$$f^{\gamma}(z)=\sum_{k=1}^{n}\gamma_k g_k(z) \qquad (1.5)$$

Note that $f'(z)$ is a polynomial of degree $n-1$. As pointed out, the derivative, normalized to be monic is then one of such convex linear combinations. The fact that the monic derivative is a convex combination of incomplete polynomials J.L.Daiz and Barrero [3] used that fact and proved the following generalization of Theorem 1.1.

Theorem 1.7. *Let $z_1, z_2, \ldots, z_n$ be n, not necessarily distinct, complex numbers. Then, the polynomial* $f^{\gamma}(z)=\sum_{k=1}^{n}\gamma_k g_k(z)$ *has all it's zeros in or on the convex hull* $K(f_n)$ *of the zeros of* $f_n(z)=\prod_{k=1}^{n}(z-z_k)$.

2. Main Results

In this paper, we extend Theorem 1.2 and Theorem 1.3 to the class of linear combination of incomplete polynomials. First, we prove the following interesting result which includes Theorem 1.7 as a special case. In fact we prove the following interesting result.

Theorem 2.1. *If all the zeros of the polynomial f (z) lie* $|z-c|\le R$ *and if w is any real or complex number satisfying the inequality*

$$|(w-c)f^{\gamma}(w)|\le|(w-c)f^{\gamma}(w)-f(w)| \qquad (2.1)$$

for every $\gamma \in \mathbb{R}_{+}^{n}$, *then* $|w-c|\le R$.

Remark **2.2.** If all the zeros of $f(z)$ lie in the $|z-c|\le R$ w is any zero of $f^{\gamma}(z)$, then $f^{\gamma}(w) = 0$, so that Inequality (2.1) is trivially satisfied. Hence by above theorem

$|w-c|\le R$. This shows that all the zeros of $f^{\gamma}(z)$ lie in $|z-c|\le R$.

Next, we prove the following result which is extension of Theorem 1.2 to the class of linear combination of incomplete polynomials which after words used to deduce the refinement of Theorem 1.7. In fact we prove the following result.

Theorem 2.3. *Let* $z_1, z_2, \ldots, z_n$, *be n, not necessarily distinct, be the zeros of f (z) with a simple zero at* z_1. *Define*

$$\omega_v=\frac{\gamma_1 z_v+(n-\gamma_1)z_1}{n}, \quad v=2,3,\ldots,n \qquad (2.2)$$

If M is an open disc or open half plane that contains z_1 *but none of the points* $\omega_2, \omega_3, \ldots, \omega_n$.

Then M is free $^{+}$ *from the zeros of the polynomial* $f^{\gamma}(z)=\sum_{k=1}^{n}\gamma_k g_k(z)$ *for every* $\gamma\in R_{+}^{n}$.

Next, we prove the following interesting result which is not only extension of Theorem 1.2 to the class of linear combination of incomplete polynomials but also a refinement of Theorem 1.7. In fact we prove the following interesting result.

Theorem 2.4. *Let f be a polynomial of degree n with zeros* $z_1, z_2, \ldots, z_n$, *then the convex hull* $K(f_n)$ *of* $n^2 - n$ *points*

$$\omega_{\nu\mu} = \frac{\gamma_1 z_\nu + (n-\gamma_1) z_\mu}{n}, \quad \nu, \mu \in \{1, 2, \ldots, n\}, \nu \neq \mu \qquad (2.3)$$

contains all the zeros of the polynomial for $f^{\gamma}(z) = \sum_{k=1}^{n} \gamma_k g_k(z)$ *every* $\gamma \in R_+^n$.

Next, we prove the following result which is extension of Theorem 1.4 to the class of linear combination of incomplete polynomials.

Theorem 2.5. *Let* $f(z)$ *be a polynomial of degree n and* $f'(w) = 0$, *for all* $\gamma \in R_+^n$, *then*

for every given real or complex number α, $f(z)$ *has atleast one zero in the region*

$$|w - \frac{\alpha + z}{2}| \leq |\frac{\alpha - z}{2}|.$$

Taking $\alpha = 0$ in Theorem 2.5 and noting that,

$$|w - \frac{z}{2}| \leq |\frac{z}{2}| \Rightarrow |w - z| \leq |z|$$

We get the following result.

Corollary 2.6. *If all the zeros of a polynomial* $f(z)$ *of degree n lie in* $D = \{z \in C : |z| \leq 1\}$, *and ω is zero of* $f'(z)$, *for all* $\gamma \in R_+^n$, *then* $f(z)$ *has at least one zero in both the Circles*

$$|w - \frac{z}{2}| \leq |\frac{1}{2}| \quad and \quad |w - z| \leq 1.$$

***Remark* 2.7.** For a n-tuple $\gamma = (1/n, 1/n, \ldots, 1/n)$, Theorem 2.5 and Corollary 2.6 reduce to Theorem 1.4 and Theorem 1.5.

3. Proof of Main Results

Proof of Theorem 2.1: Let *w* be real or complex number satisfying the inequality

$$|(w-c)f^{\gamma}(w)|\le|(w-c)f^{\gamma}(w)-f(w)| \qquad (3.1)$$

If $f(w)=0$, then clearly $|w-c|\le R$. So suppose $f(w)\neq 0$. From (3.1), we have

$$\left|\frac{(w-c)f^{\gamma}(w)}{f(w)}\right|\le\left|1-\frac{(w-c)f^{\gamma}(w)}{f(w)}\right|$$

This gives us $\operatorname{Re}\left(\frac{(w-c)f^{\gamma}(w)}{f(w)}\right)\le\frac{1}{2}$.

Let $z_1,z_2,\ldots,z_n$ be the zeros of $f(z)$, then

$$\frac{(w-c)f^{\gamma}(w)}{f(w)}=\sum_{v=1}^{n}\frac{(w-c)\gamma_v}{w-z_v}.$$

Now

$$\sum_{v=1}^{n}\operatorname{Re}\left(\frac{(w-c)}{w-z_v}\right)\gamma_v=\operatorname{Re}\sum_{v=1}^{n}\left(\frac{(w-c)}{w-z_v}\right)\gamma_v$$
$$=\operatorname{Re}\left(\frac{(w-c)f^{\gamma}(w)}{f(w)}\right)\le\frac{1}{2}.$$

Hence

$$\operatorname{Re}\left(\frac{(w-c)}{w-z_v}\right)\le\frac{1}{2\gamma_v}\le\frac{1}{2}$$

For at least one v. This gives us

$$\left|\frac{(w-c)}{(w-z_v)}\right|\le\left|1-\frac{(w-c)}{(w-z_v)}\right|=\left|\frac{(z_v-c)}{(w-z_v)}\right|$$

Hence

$$|w-c|\le|z_v-c|$$

For at least one v. Using the fact that $|z_v-c|\le R$ for all v, we get $|w-c|\le R.$

That proves the Theorem 2.1 completely. □

Proof of Theorem 2.3: If possible suppose that ζ be a zero of $f'(z)$ that lie in M , where M is an open disc or open half plane that contains z_1 but none of the points $\omega_2, \omega_3, \ldots, \omega_n$. Consider the linear transformation

$$\phi : z \to \frac{1}{\gamma_1\zeta + (n-\gamma_1)z_1 - nz} \qquad (3.2)$$

Given by

$$\phi(z) = \frac{1}{\gamma_1\zeta + (n-\gamma_1)z_1 - nz} \qquad (3.3)$$

The denominator vanishes only at $z = \frac{\gamma_1\zeta}{n} + \frac{(n-\gamma_1)z_1}{n}$, which is a point of M , since M is a convex set containing ζ and z_1. Hence ϕ maps the boundary of M onto a circle and the complement of M onto a closed unit disc D. By hypothesis $\omega_v \notin M$, which implies the point $\phi(\omega_v) \in D$ and by convexity of D, so does

$$w^* = \frac{\gamma_2\phi(w_2) + \gamma_3\phi(w_3) + \ldots + \gamma_n\phi(w_n)}{1-\gamma_1} \in D \qquad (3.4)$$

Thus there exists

$\gamma^* \notin M$

(3.5

)

such that, $\phi(\gamma^*) = \omega^*$. From by Inequality (3.4)

$$\phi(\gamma^*) = \frac{\gamma_2\phi(w_2) + \gamma_3\phi(w_3) + \ldots + \gamma_n\phi(w_n)}{1-\gamma_1}$$

$$\Rightarrow \quad (1-\gamma_1)\phi(\gamma^*) = \sum_{v=2}^{n} \gamma_v \phi(w_v).$$

This gives with the help of (3.3)

$$\frac{1-\gamma_1}{\gamma_1\zeta + (n-\gamma_1)z_1 - n\gamma^*} = \sum_{v=2}^{n} \frac{\gamma_v}{\gamma_1\zeta + (n-\gamma_1)z_1 - nw_v}$$

$$\Rightarrow \frac{1-\gamma_1}{\gamma_1\zeta+(n-\gamma_1)z_1-n\gamma^*}=\sum_{v=2}^{n}\frac{\gamma_v}{\gamma_1\zeta+(n-\gamma_1)z_1-\gamma_1 z_v-(n-\gamma_1)z_1}$$

$$\Rightarrow \frac{1-\gamma_1}{\gamma_1\zeta+(n-\gamma_1)z_1-n\gamma^*}=\sum_{v=2}^{n}\frac{\gamma_v}{\gamma_1(\zeta-z_v)}$$

$$\Rightarrow \frac{\gamma_1(1-\gamma_1)}{\gamma_1\zeta+(n-\gamma_1)z_1-n\gamma^*}=\sum_{v=2}^{n}\frac{\gamma_v}{(\zeta-z_v)} \qquad (3.6)$$

Since ζ is a zero of $f^{\gamma}(z)$, therefore we have

$$0=\frac{f^{\gamma}(\zeta)}{f(\zeta)}=\sum_{v=1}^{n}\frac{\gamma_v}{(\zeta-z_v)}=\frac{\gamma_1}{(\zeta-z_1)}+\sum_{v=2}^{n}\frac{\gamma_v}{(\zeta-z_v)}$$

Using (3.5), we obtain

$$\frac{\gamma_1}{(\zeta-z_1)}+\frac{\gamma_1(1-\gamma_1)}{\gamma_1\zeta+(n-\gamma_1)z_1-n\gamma^*}=0$$

This gives us

$$\gamma^*=\frac{\zeta+\gamma_1(n-1)z_1}{n} \qquad (3.7)$$

Since by our assumption $\zeta\in M$ and $z_1\in M$, therefore by convexity $\gamma^*\in M$.

Which is contradiction to (3.4) that $\gamma^*\notin M$. Thus our supposition is wrong. Hence we obtain the desire result.

That proves the Theorem 2.3 completely. □

Proof of Theorem 2.4: It is enough to show that any closed half plane *H* containing the points $\omega_{v\mu}$ also contains all the zeros of $f^{\gamma}(z)$.

Let $w_{v\mu}\in H$ where $v,\mu\in\{1,2,\ldots,n\}$ and $v\neq\mu\}$. If all the zeros $z_1,z_2,\ldots,z_n\in H$ then by Theorem 1.7, *H* contains all the zeros of $f^{\gamma}(z)$. But if one zero, say $z_1\notin H$, then z_1 must be a simple one, otherwise z_1 would coincide with one of the points $\omega_{v\mu}$ and can not be outside *H*.

Thus if we put $\omega_v = \omega_{\gamma 1}$ for $v = 2, 3, \ldots, n$ and $M = H$ = complement of H. By

Theorem 2.2 H^c is devoid of zeros of $f'(z)$. Hence we obtain, all the zeros of $f'(z)$ lie in

H.

That proves the Theorem 2.4 completely. □

Proof of Theorem 2.5. Let $z_1, z_2, \ldots, z_n$ be the zeros of $f(z)$ and let ω be a zero of

$f'(z)$ respectively. If $\omega = \alpha$ or $\omega = z_v$ for some $v = 1, 2, \ldots, n$, then the result is proved.

Thus, we assume that $w \neq \alpha$ and $w \neq z_v$ for any $v = 1, 2, \ldots, n$. Since ω is a zero of

$f'(z)$ and $f(w) \neq 0$, we have

$$\sum_{v=1}^{n} \frac{\gamma_v}{(w - z_v)} = \frac{f^{\gamma}(w)}{f(w)} = 0$$

This gives us

$$\sum_{v=1}^{n} \gamma_v \frac{(w - z_v) - (\alpha - z_v)}{(w - z_v)} = \sum_{v=1}^{n} \frac{\gamma_v (w - \alpha)}{(w - z_v)} = 0$$

Now, we have

$$\sum_{v=1}^{n} \frac{\gamma_v (\alpha - z_v)}{(w - z_v)} = 1$$

$$\Rightarrow \quad 1 = \sum_{v=1}^{n} \mathrm{Re} \frac{\gamma_v (\alpha - z_v)}{(w - z_v)} = \sum_{v=1}^{n} \gamma_v \ \mathrm{Re} \frac{(\alpha - z_v)}{(w - z_v)} \leq \underset{1 \leq v \leq n}{Max} \mathrm{Re} \frac{(\alpha - z_v)}{(w - z_v)}$$

This shows that for at least one $v = 1, 2, \ldots, n$

$$\mathrm{Re} \frac{(\alpha - z_v)}{(w - z_v)} \geq 1$$

Thus for at least one $v = 1, 2, \ldots, n$

$$\left| 1 - \frac{(\alpha - z_v)}{2(w - z_v)} \right| \leq \left| \frac{(\alpha - z_v)}{2(w - z)} \right|$$

From this we obtain for one $\nu = 1, 2, ..., n$

$$\left| w - \frac{(\alpha - z_\nu)}{2} \right| \leq \left| \frac{(\alpha - z_\nu)}{2} \right|.$$

That proves the Theorem 2.5 completely. □

4. Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author state that there is no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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